Int. J. Solids Structures, 1967, Vol. 3, pp. 501 to 512. Pergamon Press Ltd. Printed in Great Britain

TANGENT MODULUS THEORY FOR CYLINDRICAL SHELLS: BUCKLING UNDER INCREASING LOAD*

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Abstract—Analytical results are presented for axisymmetric plastic buckling of axially compressed cylindrical shells which is initiated under increasing load. The rate of load increase is determined by forcing the loading condition of the J_2 incremental theory of plasticity to be satisfied everywhere in the shell. Results complement those found by using classical stability concepts. The analysis constitutes the formal generalization of tangent modulus theory for plastic column behavior to axially compressed cylindrical shells.

NOTATION

E, E_T	modulus of elasticity and tangent modulus, respectively
$\dot{e}_x, \dot{e}_\theta$	extensional strain rates of middle surface in meridional and circumferential directions, respectively, equation (3)
J_2	second invariant of stress deviator
D, K	incipient extensional and flexural rigidities, respectively, equation (8)
h	shell thickness
L	length of cylinder
m	number of incipient half-waves in buckling velocity field
$\dot{M}_x, \dot{N}_\theta, \dot{N}_x$	rates of stress resultants appearing in (1) and (2); directions defined in Fig. 1
N	load at buckling
R	radius of cylinder
$N_{\tan}$	tangent modulus load, equation (17)
s_{ij}	stress deviator
v_x	incipient velocity measured along a generator
v_n	incipient velocity measured normal to middle surface, Fig. 1
T, T_1	constants in velocity field, equations (15) and (32)
x	length coordinate measured along a generator
$\dot{\varepsilon}_x, \dot{\varepsilon}_\theta$	meridional and circumferential strain rates respectively, equation (6)
$\dot{\sigma}_x, \dot{\sigma}_\theta$	rates of meridional and circumferential stresses, respectively
ξ	thickness coordinate, Fig. 1
ν	Poisson's ratio
δ_{ij}	Kronecker delta $\delta_{ij} = 1, i = j$; $\delta_{ij} = 0, i \neq j$
$\dot{\kappa}_x, \dot{\kappa}_\theta$	generalized rates of strain of middle surface associated with bending, equation (3)
$\sigma_{\tan}$	tangent modulus stress, $N_{\tan} = \sigma_{\tan} h$
λ	ratio of modulus of elasticity to tangent modulus, E/E_T
σ	stress at buckling

1. INTRODUCTION

THE present paper is concerned with the incipient axisymmetric plastic buckling of geometrically perfect axially compressed cylindrical shells which is initiated under increasing load. This is the first step in the analysis for obtaining the maximum compressive

* The research reported here was supported by the National Science Foundation through Grant GK-309 to the University of Pennsylvania.

load that the perfect system can withstand according to the principle advanced by Shanley in his well-known papers [1, 2]. However, rather than employing tedious step-by-step calculations for a shell of a particular material and following the entire load-deformation history until the maximum load is reached, the incipient results will be seen to provide a foundation from which to make a reasonable general statement about the subsequent maximum load.

Results contained herein are formally equivalent to tangent modulus theory for plastic column behavior. It is worthwhile to recall here that the proper derivation of tangent modulus theory for geometrically perfect pin-ended columns depends on the fact that at incipient bifurcation the load must increase at a rate such that at the outer fiber of the center cross-section of the column on the convex side the strain is stationary [3]. For the uniaxial state of stress in columns this ensures that further instantaneous plastic loading will occur at all other points, i.e. no points will unload elastically. However, for a combined state of stress, the minimum rate of load increase, at the load at which bifurcation is first possible, must be determined in such a manner that the particular loading criterion chosen is forced to be just satisfied in those critical points where unloading would first begin. It will be apparent from the analysis that for the simple, but reasonable, J_2 isotropic incremental theory of material behavior, satisfying the loading criterion $\dot{J}_2 > 0$ will be equivalent to imposing a stationary strain condition at certain points only for a material which is elastically incompressible. Also, it is to be especially noted that forcing the loading condition to be satisfied will eliminate the apparent indeterminacy in the incipient velocity field.

The effect of incipient membrane stretching will also be included in the analysis. The membrane stretching term appears naturally and without assumptions (other than those embodied in an engineering shell theory) in the rate equations employed. It will be shown that membrane stretching increases both the load for first bifurcation and the slope of the load–deflection curves but only by generally negligible amounts.

Using classical stability concepts, plastic buckling of axially compressed cylindrical shells was studied in detail in a recent paper [4]. By classical stability concepts is meant that buckling occurs under constant load from the idealized geometrically perfect configuration of the system. It was shown in [4] that classical concepts lead to bounds on the buckling load and, moreover, predicted quite reliably the buckling strength and the geometry of buckling for shells which buckled axisymmetrically. The results of the present study complement those of the classical stability analysis and, furthermore, constitute the exact shell theory solution to the incipient axisymmetric buckling problem without relaxing geometric constraints or making assumptions concerning the elastic–plastic interface.

The recent contributions of Hill [5], Hill and Sewell [6] and Sewell [7] to bifurcation phenomena in general and plastic buckling of columns and plates in particular should also be mentioned. In a series of papers Hill and Sewell [6] reformulated the basic plastic column buckling problem from the point of view of Hill's continuum theory of uniqueness and stability [5]. The Hill and Sewell papers have led to an increased understanding of the basic incipient column bifurcation problem.

Sewell [7] also investigated plastic plate failure from the point of view of Hill's general theory [5]. The most important conclusion reached in [7] was that the controversy surrounding deformation vs. incremental theories of plasticity, in buckling problems, is very much tied to the shape of the actual yield surface in the vicinity of the bifurcation value of stress. In this latter connection it should also be mentioned that since loading is forced to occur in the entire shell, the use of J_2 deformation theory (see [4] and [7], Part II) in this problem might again appear to be justifiable. However, Budiansky [8] has shown that in order for J_2 deformation theory to be a rigorous description of material behavior, the loading condition $\dot{J}_2 > 0$ must be abandoned and instead physical restrictions placed on the loading surface and the loading path. The limit on the allowable deviation from proportional loading is not as restrictive as the necessary introduction of a corner in the loading surface of J_2 deformation theory appearing in a special way. No attempt is made to incorporate these restrictions in the present analysis. Further attention will, therefore, be confined to the J_2 incremental theory which is always a mathematically and physically rigorous description of material behavior.

2. ANALYSIS

2.1 *Governing equations*

A rational method of investigating buckling problems is provided by the rate equation approach [9, 10]. Choosing the compressed state as the initial state and specializing the general rate equations of equilibrium for axisymmetric shells to a geometrically perfect cylindrical shell of radius R and thickness h buckling under pure axial compression, we obtain

$$\frac{\mathrm{d}^2\dot{M}_x}{\mathrm{d}x^2}+N\frac{\mathrm{d}^2v_n}{\mathrm{d}x^2}-\frac{\dot{N}_\theta}{R} = 0 \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\dot{N}_x+\frac{v_n}{R}N\right] = 0 \tag{2}$$

where dots denote the rate of change of the stress resultants measured per unit length of middle surface (Fig. 1), v_n is the incipient velocity normal to the middle surface measured positive inward toward the axis of the shell, x is the length coordinate measured along a generator of the cylinder, and $N = \sigma h$ where σ is the compressive stress at which buckling occurs. The second term in (2), which along with (1) is exact within the framework of a shell theory, is the effect of incipient circumferential stretching of the middle surface on the rate equations of equilibrium.

The circumferential membrane stretching term appears naturally in the rate equations independently of any second-order terms which may also arise because of the details of the constitutive relations eventually employed. It is not obvious that this term can be neglected compared to $\dot{N}_x$ especially since the usual case of classical buckling requires $\dot{N}_x = 0$ by definition. Furthermore, it is also important to note that, as for columns [10], the axial strain rate rigorously does not appear in (1) and (2) although for the cylindrical shell it is definitely not zero.

The incipient generalized rates of strain of the middle surface are

$$\begin{aligned} \dot{e}_\theta &= -\frac{v_n}{R} & \dot{e}_x &= \frac{\mathrm{d}v_x}{\mathrm{d}x} \\ \dot{\kappa}_x &= \frac{\mathrm{d}^2v_n}{\mathrm{d}x^2} & \dot{\kappa}_\theta &= 0 \end{aligned} \tag{3}$$

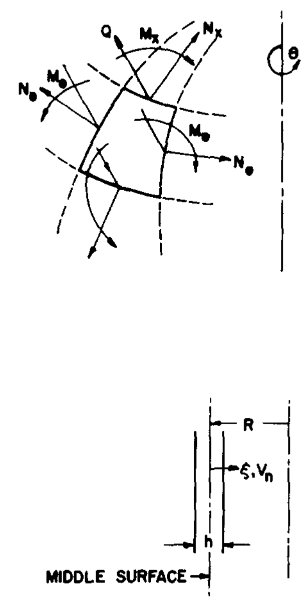

FIG. 1. Sign convention.

where v_x is the tangential velocity measured along the generatrix. We remark that $\dot{e}_x$ and $\dot{e}_\theta$ are exactly equal to the rates of change of extensions in the axial (meridional) and circumferential directions, respectively, while $\dot{\kappa}_x$ is exactly equal to the rate of change of curvature in the meridional plane, and the rate of change of curvature of the middle surface in the circumferential direction is $-\dot{e}_\theta/R$ [9].

For J_2 incremental theory, the plastic strain rate is given by

$$\begin{aligned} \dot{\varepsilon}_{ij}^P &= F(J_2)s_{ij}\dot{J}_2 \qquad \dot{J}_2 > 0 \\ \dot{\varepsilon}_{ij}^P &= 0 \qquad\qquad\quad\; \dot{J}_2 \leq 0 \end{aligned} \tag{4}$$

where

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij}$$

$$J_2 = \tfrac{1}{2}s_{ij}s_{ij}$$

and σ_{ij} is the stress tensor, δ_{ij} is the Kronecker delta and repeated indices denote summation over the range of values of the index.

Combining (4) with an isotropic linear elastic response in regions of loading, $\dot{J}_2 > 0$, we have

$$\dot{\varepsilon}_{ij} = \frac{1}{E}[(1+\nu)\dot{\sigma}_{ij} - \nu\dot{\sigma}_{kk}\delta_{ij}] + F(J_2)s_{ij}\dot{J}_2 \tag{5}$$

where E and ν are, respectively, Young's modulus and Poisson's ratio, and $F(J_2)$ can be determined from a simple tension test:

$$F(J_2) = \frac{3}{4J_2}\left[\frac{1}{E_T} - \frac{1}{E}\right]$$

E_T is the ordinary tangent modulus obtained from the uniaxial stress–strain curve.

The strain rates at a distance ξ, measured positive inward from the middle surface, are given by the usual expressions

$$\dot{\varepsilon}_x = \dot{e}_x - \xi\dot{\kappa}_x, \qquad \dot{\varepsilon}_\theta = \dot{e}_\theta \tag{6}$$

Consistent with the assumption of the preservation of the normal element, the first of (6) is exact while the second of (6) is approximate and is obtained by neglecting the small term ξ/R compared to unity in the exact expression [11].

For the shell, when buckling commences from a state of uniaxial compression ($\sigma_x = -\sigma$), the stress rates in regions of loading are

$$\begin{aligned} \dot{\sigma}_x &= \frac{E}{(5-4\nu)\lambda-(1-2\nu)^2}[(\lambda+3)\dot{\varepsilon}_x+2(\lambda-1+2\nu)\dot{\varepsilon}_\theta] \\ \dot{\sigma}_\theta &= \frac{E}{(5-4\nu)\lambda-(1-2\nu)^2}[4\lambda\dot{\varepsilon}_\theta+2(\lambda-1+2\nu)\dot{\varepsilon}_x] \end{aligned} \tag{7}$$

where $\lambda = E/E_T$.

Substituting (6) and (7) into appropriate expressions for the stress rate resultants, and integrating through the thickness of the shell, it can be shown [4] that in regions of loading

$$\begin{aligned} \dot{N}_\theta &= D\left[\lambda\dot{e}_\theta+\frac{(2\nu+\lambda-1)}{2}\dot{e}_x\right] \\ \dot{N}_x &= D\left[\frac{\lambda+3}{4}\dot{e}_x+\frac{(2\nu+\lambda-1)}{2}\dot{e}_\theta\right] \\ \dot{M}_\theta &= \frac{K(2\nu+\lambda-1)}{2}\dot{\kappa}_x \\ \dot{M}_x &= \frac{K(\lambda+3)}{4}\dot{\kappa}_x \end{aligned} \tag{8}$$

where

$$D = \frac{4Eh}{(5-4\nu)\lambda-(1-2\nu)^2}$$

$$K = \frac{Eh^3}{3[(5-4\nu)\lambda-(1-2\nu)^2]}$$

It is necessary to remark here that the expressions given for $\dot{M}_x$ and $\dot{N}_x$ are obtained by neglecting the small term ξ/R compared to unity when integrating through the shell thickness, i.e. the trapezoidal shape of an element is ignored. This assumption combined with that embodied in the second of (6) is equivalent to neglecting small terms containing $\dot{e}_x$, $\dot{e}_\theta$ and $\dot{\kappa}_x$ in the generalized stress rate–generalized strain rate relations for the shell material. These combined assumptions are, of course, the ones usually made in the theory of thin elastic shells [11].

Equations (8) in their present form are used throughout the paper. The membrane stretching effect then only refers to whether or not the second term in equation (2) is retained in the analysis. In this manner it is possible to focus attention on, as well as to evaluate, the effect of changes in geometry caused by buckling on the rate equations of equilibrium apart from any other assumptions.

For the state of stress under consideration, $\dot{J}_2 > 0$ implies that at buckling

$$\dot{\sigma}_\theta - 2\dot{\sigma}_x > 0 \tag{9}$$

Using (7), and the fact that $\lambda > 1$ and theoretically $-1 \leq \nu \leq \frac{1}{2}$, it follows that in terms of strain rates at buckling (9) is equivalent to

$$(1-2\nu)\dot{\varepsilon}_\theta - (2-\nu)\dot{\varepsilon}_x > 0 \tag{10}$$

Regardless of the details of the analysis, it is obvious from (10) that if the material is elastically incompressible, $\nu = \frac{1}{2}$, forcing the loading condition to be just satisfied at certain critical points will be equivalent to imposing the condition that the axial strain be stationary at those points.

2.2 *Neglecting membrane stretching*

When membrane stretching is neglected, equation (2) is replaced by

$$\frac{\mathrm{d}\dot{N}_x}{\mathrm{d}x} = 0 \tag{11}$$

or the equivalent expression

$$\dot{N}_x = \text{constant} = -\dot{N} = -\dot{\sigma}h \tag{12}$$

where $\dot{N}$ is the rate of change of the applied compressive load. Classical buckling means $\dot{N} = 0$.

Substituting (3, 8, 10) into (1) leads to the following equation:

$$\frac{K(\lambda+3)}{4}\frac{\mathrm{d}^4 v_n}{\mathrm{d}x^4} + N\frac{\mathrm{d}^2 v_n}{\mathrm{d}x^2} + \frac{4Ehv_n}{(\lambda+3)R^2} = \frac{-2(2\nu+\lambda-1)}{\lambda+3}\frac{\dot{N}}{R} \tag{13}$$

The particular solution to (13) is

$$v_n^p = -\frac{(2\nu+\lambda-1)}{2E}R\dot{\sigma} \tag{14}$$

which is the velocity of uniform outward expansion of the shell due to the compressive stress rate $\dot{\sigma}$.

The complementary solution to (13), v_n^c, is the non-uniform velocity field at bifurcation. For a simply supported cylinder of length L

$$x = 0, x = L; \qquad v_n^c = 0, \dot{M}_x = 0$$

it was found [4] that

$$v_n^c = T\sin\frac{m_{\tan}\pi}{L}x \tag{15}$$

where T is a constant to be determined and $m_{\tan}$ is an integer defined by

$$\frac{m_{\tan}\pi}{L} = \frac{2}{[hR(\lambda+3)]^{\frac{1}{2}}}\{3[(5-4\nu)\lambda-(1-2\nu)^2]\}^{\frac{1}{4}} \tag{16}$$

The critical load under which (15) and (16) occurs is given by

$$N_{\tan} = \sigma_{\tan} h = \frac{2Eh^2}{R} \frac{1}{\{3[(5-4\nu)\lambda-(1-2\nu)^2]\}^{\frac{1}{2}}} \tag{17}$$

It is important to recall that using classical concepts (15, 16 and 17) was shown to be the solution for a very useful model of an actual cylinder and it was concluded that (17) was a lower bound to the actual buckling load. In addition, a very minor restriction on the tangent modulus at buckling appeared and it was not possible to determine the multiplicative constant in the velocity field [4]. The present solution removes all previous restrictions and provides a rigorous independent proof that (17) is a lower bound to the buckling load for a perfect cylinder.

Since equations (15–17) were derived on the basis that $\dot{J}_2 > 0$ everywhere it remains to force this condition to be true. In terms of the solution, $\dot{J}_2 > 0$ implies

$$-\frac{2Ev_n^c}{(5-4\nu)\lambda-(1-2\nu)^2}\left[\frac{(5-4\nu)\lambda-(1-2\nu)^2}{(\lambda+3)R}+\xi(2-\nu)\left(\frac{m_{\tan}\pi}{L}\right)^2\right]+\dot{\sigma} > 0 \tag{18}$$

Note in (18) that if the loading condition is satisfied at the critical locations when $v_n^c = T$ and $\xi = h/2$ (inner surface at center of inward wave) then it will be satisfied everywhere. Hence, setting $\dot{J}_2 = 0$ at the above critical locations and using (16) and (17) results in

$$T = \frac{R\dot{\sigma}(\lambda+3)}{2E}\left\{\frac{1}{1+3(2-\nu)\{\sigma_{\tan}/[E(h/R)]\}}\right\} \tag{19}$$

The total incipient velocity field from the pre-buckled compressed shape is thus uniquely given by

$$v_n = \frac{R\dot{\sigma}}{2E}\left\{\frac{(\lambda+3)\sin[(m_{\tan}\pi x)/L]}{1+3(2-\nu)\{\sigma_{\tan}/[E(h/R)]\}}-(2\nu+\lambda-1)\right\} \tag{20}$$

It is interesting that v_n is purely outward ($v_n < 0$) if buckling occurs at a value of $\lambda > \lambda_0$ where

$$\lambda_0 = \frac{11-16\nu+[(11-16\nu)^2-48(1-2\nu)^2]^{\frac{1}{2}}}{6} \tag{21}$$

or for

$$\nu = 0{\cdot}5 \qquad \lambda_0 = 1$$

$$\nu = 0{\cdot}33 \qquad \lambda_0 = 1{\cdot}82$$

$$\nu = 0 \qquad \lambda_0 = 3{\cdot}26$$

However, the outward nature of the total v_n is not what is meant by outward buckling observed in tests. Rather, outward buckling usually means that when the uniform outward movement due to radial expansion is subtracted from the total displacement pattern, outward waves predominate over inward waves. Or equivalently, if the increments of the waveform displacement from each successive state are summed, the shell will show a pronounced outward movement. In this sense the results of [4] give more a picture of the final geometric pattern to be expected in tests while the present results indicate no preference for outward or inward buckling. The reason for this is that no account has been taken of unloading. Since the centers of the inward waves at the inner surface of the shell are getting ready to unload, this will increase their stiffness compared to the stiffness of the outer waves. If an analysis was carried out for additional load increments beyond the incipient state, it would have to show that outward waves grow at a faster rate than inward waves.

If we let u_1 be the wave form displacement measured positive inward, then initially

$$v_n^c = \dot{u}_1 \tag{22}$$

Hence, it follows from (15, 19 and 22) that

$$v_{n\,\max}^c = T = \dot{u}_{1c} = \frac{R\dot{\sigma}(\lambda+3)}{2E}\left\{\frac{1}{1+3(2-\nu)\{\sigma_{\tan}/[E(h/R)]\}}\right\} \tag{23}$$

where $\dot{u}_{1c}$ is the rate of displacement at the center of the inward wave. Using (17) we can rewrite (23) as

$$\frac{h}{\sigma_{\tan}}\frac{\dot{\sigma}}{\dot{u}_{1c}} = \frac{h}{\sigma_{\tan}}\frac{\mathrm{d}\sigma}{\mathrm{d}u_{1c}} = \frac{1}{\lambda+3}[\{3[(5-4\nu)\lambda+(1-2\nu)^2]\}^{\frac{1}{2}}+6(2-\nu)] \tag{24}$$

It is immediately obvious from (24) that the applied load must increase beyond $\sigma_{\tan}$ in order to continue the deformation. Furthermore, the right hand side of (24) decreases with increasing λ and very quickly becomes less than unity. It is also evident, from setting $\dot{J}_2 = 0$ at $v_n^c = T$ and $\xi = h/2$, that equation (24) gives the minimum incipient slope of the load–deflection curve emanating from $\sigma_{\tan}$, equation (17).

The analogous result to (24) for typical column sections is [3]

$$\frac{h}{\sigma_{\tan}}\frac{\mathrm{d}\sigma}{\mathrm{d}u_c} = \begin{cases} 8 & \text{Circular} \\ 6 & \text{Rectangular} \\ 2 & \text{Idealized H-section} \end{cases} \tag{25}$$

where u_c is the lateral deflection at the column center and h is the distance between extreme fibers in bending.

Figure 2 is a typical qualitative representation of known load–deflection behavior for columns [3] and conjectured behavior for cylindrical shells which buckle axisymmetrically. The curve for the shell was drawn as being typical by comparing the initial slopes, (24) to (25), coupled with the results of carefully performed experiments [4]. In addition, one could also argue without making detailed calculations that the expression for the slope at the next load increment beyond $\sigma_{\tan}$ can not be radically different than (24) because only a very small portion of the inward waves would have unloaded and the geometry would not have appreciably changed. Since for real materials, E_t decreases (λ increases) as the average stress increases, (24) is an indication that the slope of the load–deflection curve would continue to decrease. Note that for columns, where numerical calculations *do show* that the slope is continually decreasing (large deflection effects ignored), one cannot use this argument since the right side of (25) is independent of the tangent modulus at buckling.

Hence, since even for perfect columns the maximum load is not very much larger than the tangent modulus load [3], it is concluded that for geometrically perfect axially compressed cylindrical shells tedious step-by-step calculations would show that there is a negligible difference between the maximum load and $N_{\tan}$. Of course, the actual difference depends on the details of the uniaxial stress-strain curve but for real materials the difference will be very small.

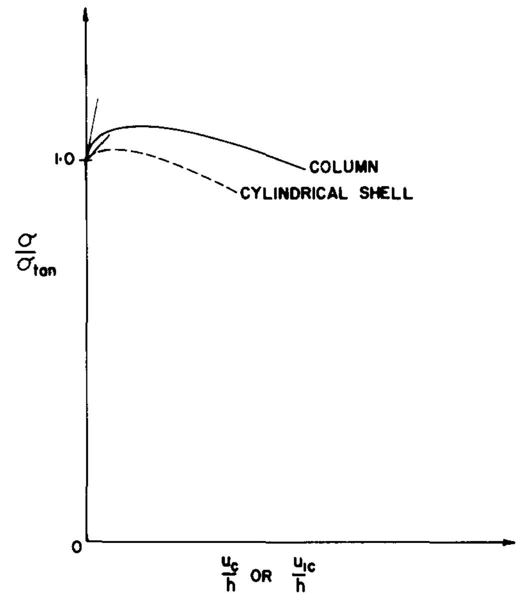

FIG. 2. Qualitative load–deflection behavior.

2.3 *Membrane stretching included*

Some caution must be exercised before equation (2) is used in an analysis. Integrating (2) we have

$$\dot{N}_x - \dot{e}_\theta N = \text{constant} \tag{26}$$

When the shell does not buckle $\dot{N}_x = -\dot{N}$ although $\dot{e}_\theta$ is clearly not zero. Hence the constant in (26) is not always simply equal to the rate of change of the applied load. It is not difficult to show, however, that in the reduced form of (26) only the nonhomogeneous part of the velocity field enters as follows

$$\dot{N}_x - \dot{e}_{\theta 1} N = -\dot{N} = -\dot{\sigma} h \tag{27}$$

where

$$\dot{e}_{\theta 1} = -\frac{v_{n1}}{R}$$

and v_{n1} is the buckling velocity to be determined.

To facilitate the solution let

$$\dot{N}_x = -\dot{\sigma} h + \dot{N}_{x1}$$

where $\dot{N}_{x1}$ is the change due to membrane stretching. Hence (27) becomes

$$\dot{N}_{x1} - \dot{e}_{\theta 1} N = 0 \tag{28}$$

Combining (1, 3, 8 and 28) the differential equation governing the buckling velocity is found to be

$$\frac{K(\lambda+3)}{4}\frac{\mathrm{d}^4 v_{n1}}{\mathrm{d}x^4}+N\frac{\mathrm{d}^2 v_{n1}}{\mathrm{d}x^2}+\frac{4Eh}{(\lambda+3)R^2}\left[1+\frac{\sigma}{2E}(2\nu+\lambda-1)\right]v_{n1} = 0. \tag{29}$$

Of course, if interest was in the total incipient velocity field, v_n^P given by (14) would be added to v_{n1}. It is immediately apparent that the entire membrane stretching effect is contained in the last term of (29). Furthermore, for the classical elastic case, $\lambda = 1$, it is obvious that the membrane effect will be negligible. However, it is not obvious that when buckling occurs at a sufficiently small value of tangent modulus (λ large), that the membrane effect will be small enough to be neglected.

Solving the eigenvalue problem posed by equation (29) and the boundary conditions for a simply supported shell, we find

$$\frac{\sigma}{\sigma_{\tan}} = \frac{2\nu+\lambda-1}{2R}\left(\frac{K}{Eh}\right)^{\frac{1}{2}}+\left\{\left(\frac{2\nu+\lambda-1}{2R}\right)^2\frac{K}{Eh}+1\right\}^{\frac{1}{2}} \tag{30}$$

$$\frac{m\pi}{L} = \frac{m_{\tan}\pi}{L}\left[1+\frac{\sigma}{2E}(2\nu+\lambda-1)\right]^{\frac{1}{4}} \tag{31}$$

$$v_{n1} = T_1 \sin\frac{m\pi}{L}x \tag{32}$$

where T_1 is a constant to be determined and $m_{\tan}\pi/L$ and $\sigma_{\tan}$ are given, respectively, by (16) and (17).

Although (30) is exact, it is cumbersome. The following are useful approximations to (30):
To terms of order $(h/R)^2$,

$$\frac{\sigma}{\sigma_{\tan}} = 1+\left(\frac{2\nu+\lambda-1}{2}\right)\frac{\sigma_{\tan}}{2E}\left[1+\left(\frac{2\nu+\lambda-1}{2}\right)\frac{\sigma_{\tan}}{4E}\right] \tag{33}$$

To terms of order h/R,

$$\frac{\sigma}{\sigma_{\tan}} = 1+\left(\frac{2\nu+\lambda-1}{2}\right)\frac{\sigma_{\tan}}{2E}. \tag{34}$$

In order to obtain an idea of the orders of magnitude involved, equation (34) is shown below in a reduced form for $\nu = \frac{1}{2}$

$$\sigma = \left(1+\frac{1}{6}\frac{h}{R}\lambda^{\frac{1}{2}}\right)\sigma_{\tan}. \tag{35}$$

A representative worst value for the coefficient of $\sigma_{\tan}$ in (35) would be when, say, $\lambda = 100$ and $h/R = \frac{1}{10}$. Hence

$$\sigma = 1{\cdot}167\sigma_{\tan}. \tag{36}$$

However, using equation (17) to predict the stress for first bifurcation, instead of (36), does not mean that the error involved will be on the order of 17 per cent. In fact, using (36) will lead to a very slight, almost undetectable, increase in the buckling prediction. It must be emphasized that the actual difference between (36) and (17) will be almost wiped out because of the large changes in tangent modulus due to small increases in stress. Precisely this same effect is responsible for the closeness of reduced modulus and tangent modulus predictions for columns and cylindrical shells [4] and for the fact that columns exhibit a maximum load instead of approaching the reduced modulus load asymptotically [2, 3].

Forcing the loading condition to be satisfied leads to

$$T_1 = \frac{R\dot{\sigma}(\lambda+3)}{2E}\left\{\frac{1}{1+3(2-\nu)\{\sigma_{\tan}/[E(h/R)]\}+(2-\nu)\sigma/E[\frac{3}{4}(2\nu+\lambda-1)\{\sigma_{\tan}/[E(h/R)]\}-1]}\right\} \tag{37}$$

where the effect of membrane stretching is exhibited in the last term of the denominator of (37). In order to arrive at the form (37) it was necessary to use

$$\left[1+\frac{\sigma}{2E}(2\nu+\lambda-1)\right]^{\frac{1}{2}} \approx 1+\frac{\sigma}{4E}(2\nu+\lambda-1) \tag{38}$$

which is sufficiently accurate for the purposes of this discussion.

It is not difficult to show that the membrane effect is quite small, in general, due to the presence of the coefficient σ/E. Moreover, if buckling occurs at a value of λ such that

$$\tfrac{3}{4}(2\nu+\lambda-1)\frac{\sigma_{\tan}}{E(h/R)} > 1 \tag{39}$$

the initial slope of the load–deflection curve computed from (22) and (37) will be larger, although only very slightly so, than that given by equation (24) for the same value of λ. For completeness, inequality (39) leads to

$$\lambda > \frac{13-14\nu+[(13-14\nu)^2-21(1-2\nu)^2]^{\frac{1}{2}}}{3} = \begin{cases} 4 & \nu = 0{\cdot}5 \\ 5{\cdot}5 & \nu = 0{\cdot}33 \\ 8{\cdot}4 & \nu = 0 \end{cases} \tag{40}$$

Hence, the preceding calculations show that the membrane stretching effect is indeed generally small and can be neglected in the rate equations of equilibrium.

3. CONCLUDING REMARKS

The analysis has shown that bifurcation of equilibrium positions can begin at the load predicted by (17) but under increasing load, equation (24). To make the definite statement that bifurcation does begin at (17) probably requires the use of either a Duberg and Wilder [3] type approach, i.e. computing load–deflection curves for imperfect systems and allowing the degree of imperfection to go to zero, or perhaps awaits the development of as yet undiscovered minimum principles.

It is worthwhile to mention again that (1) forcing the loading condition to be satisfied removes the indeterminacy in the velocity field of the classical analysis; (2) the difference between the maximum load that a perfect system can withstand and the load predicted by equation (17) is likely to be negligible; and (3) for real materials the effect of incipient circumferential membrane stretching on the load for first bifurcation and on the slope of the load–deflection curve can be neglected.

Results presented herein are the complete and formal generalization of tangent modulus theory for columns to axially compressed cylindrical shells which buckle axisymmetrically.

REFERENCES

[1] F. R. SHANLEY, The column paradox. *J. aeronaut. Sci.* **13**, 678 (1946).
[2] F. R. SHANLEY, Inelastic column theory. *J. aeronaut. Sci.* **14**, 261–268 (1947).
[3] J. E. DUBERG and T. W. WILDER, III. Inelastic column behavior. NACA Rept. 1072 (1952).
[4] S. C. BATTERMAN, Plastic buckling of axially compressed cylindrical shells. *AIAA Jnl* **3**, 316–325 (1965).
[5] R. HILL, Some basic principles in the mechanics of solids without a natural time. *J. Mech. Phys. Solids* **7**, 209–225 (1959).
[6] R. HILL and M. J. SEWELL, A General theory of inelastic column failure. Part I, *J. Mech. Phys. Solids* **8**, 105–111 (1960); Part II, **8**, 112–118 (1960); Part III, **10**, 285–300 (1962).
[7] M. J. SEWELL, A general theory of elastic and inelastic plate failure. Part I, *J. Mech. Phys. Solids*, **11**, 377–393 (1963); Part II, **12**, 279–297 (1964).
[8] B. BUDIANSKY, A reassessment of deformation theories of plasticity. *J. appl. Mech.* **26**, 259–264 (1959).
[9] S. C. BATTERMAN, Load-deformation behavior of shells of revolution. *Proc. Am. Soc. civ. Engrs J. Engng Mech. Div.* **90**, EM6, 1–19 (1964).
[10] S. C. BATTERMAN, On rate equations for plane curved beams. *J. appl. Mech.* To be published.
[11] W. FLUGGE, *Stresses in Shells*, chapter 5. Springer-Verlag, (1960).

(*Received* 5 *July* 1966; *revised* 28 *November* 1966)

Résumé—Des résultats analytiques sont présentés relativement au flambage plastique symétrique à l'axe d'enveloppes cylindriques comprimées axialement ayant lieu sous des charges croissantes. Le taux d'accroissement de la charge est déterminé en forcant la condition de charge des accroissements J_2 de la théorie de la plasticité à être satisfaite en tout point de l'enveloppe. Les résultats complètent sont trouvés en employant les concepts de stabilité classique. L'analyse constitue la généralisation formelle de la théorie du module tangentiel pour le comportement d'une colonne en plastique envers des enveloppes cylindriques comprimées axialement.

Zusammenfassung—Analytische Resultate werden gegeben für die achsensymmetrische plastische Knickung in der Achsenrichtung gedrückter Zylinderschalen bei ansteigender Belastung. Die Zunahme der Belastung wird bestimmt, indem man den Belastungszustand J_2 der zunehmenden Plastizitätstheorie an jeder Stelle der Schale befriedigt. Die Resultate sind komplementär mit denen die mittels klassischer Stabilitätskonzepte gewonnen wurden. Die Analyse bildet die formelle Verallgemeinerung der Tangentenmodultheorie für das Verhalten plastischer Säulen für zylindrische Schalen.

Абстракт—Приводятся аналитические результаты исследования осесимметричного выпучивания цилиндрической оболочки, нагруженной осевой силой, вызываемого приростом нагрузки. Скорость прироста нагрузки задана так, что во всей оболочке выполнено условие нагружения в смысле J_2, по теории пластического течения. Результаты дополняют относительные результаты, основанные на классических представлениях об устойчивости. Расчеты дают возможность формально обобщить теорию касательного модула в поведении пластических колонн на цилиндрические оболочки, сжатые осевой силой.